

FRACTIONAL INTEGRATION AND FRACTIONAL DIFFERENTIATION FOR d -DIMENSIONAL JACOBI EXPANSIONS

CRISTINA BALDERRAMA^{A,*} AND WILFREDO O. URBINA^{R^{A,B}}.

^a *Departamento de Matemáticas, Facultad de Ciencias, UCV. Apartado 40009, Los Chaguaramos, Caracas 1041-A Venezuela*

^b *Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA*

ABSTRACT. In this paper we consider an alternative orthogonal decomposition of the space L^2 associated to the d -dimensional Jacobi measure in order to obtain an analogous result to P.A. Meyer's Multipliers Theorem for d -dimensional Jacobi expansions. Then we define and study the Fractional Integral, the Fractional Derivative and the Bessel potentials induced by the Jacobi operator. We also obtain a characterization of the Sobolev or potential spaces and a version of Calderón's reproduction formula for the d -dimensional Jacobi measure.

RÉSUMÉ. Dans cet article nous considérons une décomposition orthogonale alternative de l'espace L^2 associée à la mesure de Jacobi d -dimensionnelle afin d'obtenir de résultat analogues au Théorème des Multiplicateurs de P.A. Meyer pour les développements d -dimensionnels de Jacobi. Nous définissons et étudions l'intégral Fractionnaire, la dérivée Fractionnaire et les potentiels de Bessel induits par l'opérateur de Jacobi. Nous obtenons également une caractérisation des espaces de Sobolev ou potetiel de Jacobi et une version de la formule de reproduction de Calderón pour la mesure de Jacobi d -dimensionnelle.

* Corresponding author

E-mail addresses: cbalde@euler.ciens.ucv.ve (C. Balderrama), wurbina@euler.ciens.ucv.ve (W. Urbina).

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1. INTRODUCTION

For the parameters $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \dots, \beta_d)$, in \mathbb{R}^d $\alpha_i, \beta_i > -1$ let us consider the (normalized) Jacobi measure on $[-1, 1]^d$ defined as

$$(1) \quad \mu_{\alpha, \beta}^d(dx) = \prod_{i=1}^d \frac{1}{2^{\alpha_i + \beta_i + 1} B(\alpha_i + 1, \beta_i + 1)} (1 - x_i)^{\alpha_i} (1 + x_i)^{\beta_i} dx_i.$$

This normalization gives a probability measure. It is not usually considered in classical orthogonal polynomial theory.

The d -dimensional Jacobi operator is given by

$$(2) \quad \mathcal{L}^{\alpha, \beta} = \sum_{i=1}^d \left[(1 - x_i^2) \frac{\partial^2}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2) x_i) \frac{\partial}{\partial x_i} \right],$$

It is not difficult to see that this is formally a symmetric operator in the space $L^2([-1, 1]^d, \mu_{\alpha, \beta}^d)$.

For a multi-index $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{N}^d$ let $\vec{p}_{\kappa}^{\alpha, \beta}$ be the normalized Jacobi polynomial of order κ , defined for $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ as

$$\vec{p}_{\kappa}^{\alpha, \beta}(x) = \prod_{i=1}^d p_{\kappa_i}^{\alpha_i, \beta_i}(x_i)$$

where $p_n^{\alpha, \beta}$ for $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > -1$, is the one dimensional normalized Jacobi polynomial that can be defined using Rodrigues formula (see [13]),

$$(1 - x)^{\alpha} (1 + x)^{\beta} c_n p_n^{\alpha, \beta}(x) = \frac{(-1)^n}{2^n n!} \frac{\partial^n}{\partial x^n} \{ (1 - x)^{\alpha+n} (1 + x)^{\beta+n} \}, x \in [-1, 1].$$

As the one dimensional Jacobi polynomials are orthonormal with respect to the one dimensional (normalized) Jacobi measure on $[-1, 1]$

$$\mu_{\alpha, \beta}(dx) = \frac{1}{2^{\alpha + \beta + 1} B(\alpha + 1, \beta + 1)} (1 - x)^{\alpha} (1 + x)^{\beta} dx,$$

it is immediate that the normalized Jacobi polynomials $\{\vec{p}_{\kappa}^{\alpha, \beta}\}$ are orthonormal with respect to the d -dimensional Jacobi measure. Moreover the family $\{\vec{p}_{\kappa}^{\alpha, \beta}\}$ is an orthonormal Hilbert basis of $L^2([-1, 1]^d, \mu_{\alpha, \beta}^d)$.

It is well known that Jacobi polynomials are eigenfunctions of the Jacobi operator $\mathcal{L}^{\alpha, \beta}$ with eigenvalue $-\lambda_{\kappa} = -\sum_{i=1}^d \kappa_i (\kappa_i + \alpha_i + \beta_i + 1)$, that is,

$$(3) \quad \mathcal{L}^{\alpha, \beta} \vec{p}_{\kappa}^{\alpha, \beta} = -\lambda_{\kappa} \vec{p}_{\kappa}^{\alpha, \beta}$$

The d -dimensional Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t \geq 0}$ is the Markov operator semigroup in $L^2([-1, 1]^d, \mu_{\alpha,\beta}^d)$ associated to the Markov probability kernel semigroup (see [2],[5] or [15]).

$$P^{\alpha,\beta}(t, x, dy) = \sum_{\kappa \in \mathbb{N}^d} e^{-\lambda_\kappa t} \vec{p}_\kappa^{\alpha,\beta}(x) \vec{p}_\kappa^{\alpha,\beta}(y) \mu_{\alpha,\beta}^d(dy) = p_d^{\alpha,\beta}(t, x, y) \mu_{\alpha,\beta}^d(dy),$$

that is

$$T_t^{\alpha,\beta} f(x) = \int_{[-1,1]^d} f(y) P^{\alpha,\beta}(t, x, dy) = \int_{[-1,1]^d} f(y) p_d^{\alpha,\beta}(t, x, y) \mu_{\alpha,\beta}^d(dy).$$

Unfortunately, there is not a reasonable explicit representation of the kernel $P^{\alpha,\beta}(t, x, dy)$, but that is not needed in what follows. Alternatively the d -dimensional Jacobi semigroup can be defined as the tensorization of one dimensional Jacobi semigroups (cf. [15]).

The d -dimensional Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t \geq 0}$ is a Markov diffusion semigroup, conservative, symmetric, strongly continuous on $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ of positive contractions on L^p , with infinitesimal generator $\mathcal{L}^{\alpha,\beta}$. By (3) we have

$$(4) \quad T_t^{\alpha,\beta} \vec{p}_\kappa^{\alpha,\beta} = e^{-\lambda_\kappa t} \vec{p}_\kappa^{\alpha,\beta}.$$

It can be proven that for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{R}^d$ with $\alpha_i, \beta_i > -\frac{1}{2}$, $\{T_t^{\alpha,\beta}\}_{t \geq 0}$ is not only a contraction on $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ but it is also an hypercontractive semigroup, that is to say, for any initial condition $1 < q(0) < \infty$ there exists an increasing function $q : \mathbb{R}^+ \rightarrow [q(0), \infty)$, such that for every f and all $t \geq 0$,

$$\|T_t^{\alpha,\beta} f\|_{q(t)} \leq \|f\|_{q(0)}.$$

The proof of this fact is not very well known and it is an indirect one. It is based on the fact that the one dimensional Jacobi operator satisfies a Sobolev inequality, that can be proved by checking that it satisfies a curvature-dimension inequality, this result was obtained by D. Bakry in [4]. Then it can be proved that this implies a logarithmic Sobolev inequality for the one dimensional Jacobi operator. As the logarithmic Sobolev inequality is stable under tensorization, [1], we have that the d -dimensional Jacobi operator also satisfies a logarithmic Sobolev inequality and then using L. Gross' famous result [10], that asserts the equivalence between the hypercontractivity property and the logarithmic Sobolev inequality, the result is obtained. All the implications between these functional inequalities and L. Gross' result can be found in [1]. A detailed proof of the hypercontractivity property for the Jacobi semigroup can be found in [2], see also [15].

From now on we will consider only the Jacobi semigroup for the parameters $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{R}^d$ with $\alpha_i, \beta_i > -\frac{1}{2}$.

For $0 < \delta < 1$ we define the generalized d -dimensional Poisson–Jacobi semigroup of order δ , $\{P_t^{\alpha, \beta, \delta}\}$, as

$$(5) \quad P_t^{\alpha, \beta, \delta} f(x) = \int_0^\infty T_s^{\alpha, \beta} f(x) \mu_t^\delta(ds).$$

where $\{\mu_t^\delta\}$ are the stable measures on $[0, \infty)$ of order δ .^(*) The generalized d -dimensional Poisson–Jacobi semigroup of order δ is a strongly continuous semigroup on $L^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$ with infinitesimal generator $(-\mathcal{L}^{\alpha, \beta})^\delta$. Again, by (3) we have that

$$(6) \quad P_t^{\alpha, \beta, \delta} \vec{p}_\kappa^{\alpha, \beta} = e^{-\lambda_\kappa^\delta t} \vec{p}_\kappa^{\alpha, \beta}.$$

In the particular case $\delta = 1/2$, we have the d -dimensional Poisson–Jacobi semigroup. As it is relevant in what follows we will denote it simply by $P_t^{\alpha, \beta} = P_t^{\alpha, \beta, 1/2}$. In this case we can explicitly compute $\mu_t^{1/2}$,

$$\mu_t^{1/2}(ds) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds$$

and we have Bochner’s subordination formula,

$$(7) \quad P_t^{\alpha, \beta} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u}^{\alpha, \beta} f(x) du.$$

The paper is organized as follows. In the next section we will give a decomposition of the space $L^2([-1, 1]^d, \mu_{\alpha, \beta}^d)$, that we call a modified Wiener–Jacobi decomposition. In section 3, using this decomposition and the hypercontractivity property of the d -dimensional Jacobi semigroup, we present an analogous of Meyer’s Multiplier Theorem, [12], for d -dimensional Jacobi expansions and define and study, as in the one dimensional case, [3], the fractional derivatives the fractional integrals, the Bessel potentials for the d -dimensional Jacobi operator, and Jacobi Sobolev spaces associated to the d -dimensional Jacobi measure. Finally we also study the asymptotic behavior of the d -dimensional Poisson–Jacobi semigroup and, as a consequence, we present a version of Calderon’s reproducing formula.

For others expansions in terms of classical orthogonal polynomials there have been similar notions. In [11] it was studied the fractional derivative for the Gaussian

(*) Stable measures on $[0, \infty)$ are Borel measures on $[0, \infty)$ such that its Laplace transform verify $\int_0^\infty e^{-\lambda s} \mu_t^\delta(ds) = e^{-\lambda^\delta t}$. For δ fixed, $\{\mu_t^\delta\}$ form a semigroup with respect to the convolution operation, see [8].

measure, that is, in the case of Hermite polynomial expansions. In this article they also obtain characterizations of the Gaussian Sobolev spaces and a version of Calderon's reproducing formula for the Gaussian measure.

In [9] the Laguerre polynomial expansions was studied. In this article the authors also obtain an analogous of P.A. Meyer's Multiplier Theorem for Laguerre expansions and introduce fractional derivatives and fractional integrals in this setting. They also study different Sobolev spaces associated to Laguerre expansions and Riesz-Laguerre transforms.

Thus with this paper we complete the study of these notions for classical orthogonal polynomials. In [3] we have studied the one dimensional case for Jacobi expansions and in the present article we extend this notions to higher dimensions. Contrary to the Hermite and Laguerre cases, in this case the passage from one dimension to several dimensions is not straight forward, due to the no linearity of the eigenvalues of the Jacobi operator. This will be explained with more detail in the next section.

In order to simplify notation, we will always not explicitly refer the dependency of the dimension d . For instance, we will denote by $\|\cdot\|_p$ the norm in $L^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$, that is, without explicitly referring the dependency of the dimension d .

2. A MODIFIED WIENER-JACOBI DECOMPOSITION.

Let us consider for each $n \geq 0$, $C_n^{\alpha, \beta}$ is the closed subspace of $L^2([-1, 1]^d, \mu_{\alpha, \beta}^d)$ generated by the linear combinations of $\{\vec{p}_\kappa^{\alpha, \beta} : |\kappa| = n\}$, where, as usual for a multi-index κ , $|\kappa| = \sum_{i=1}^d \kappa_i$. Since $\{\vec{p}_\kappa^{\alpha, \beta}\}$ is an orthonormal basis of $L^2([-1, 1]^d, \mu_{\alpha, \beta}^d)$, we have the orthogonal decomposition

$$(8) \quad L^2([-1, 1]^d, \mu_{\alpha, \beta}^d) = \bigoplus_{n=0}^{\infty} C_n^{\alpha, \beta}.$$

This is the Wiener-Jacobi decomposition of $L^2([-1, 1]^d, \mu_{\alpha, \beta}^d)$, which is analogous to the Wiener decomposition of $L^2(\mathbb{R}^d, \gamma_d)$ is the Gaussian case.

For $f \in L^2([-1, 1]^d, \mu_{\alpha, \beta}^d)$, its Jacobi expansion is given by

$$f = \sum_{n=0}^{\infty} \sum_{|\kappa|=n} \hat{f}(\kappa) \vec{p}_\kappa^{\alpha, \beta},$$

with $\hat{f}(\kappa) = \int_{[-1,1]^d} f(y) \vec{p}_\kappa^{\alpha,\beta} \mu_{\alpha,\beta}^d(dy)$. Then we have the following spectral decompositions

$$\begin{aligned}\mathcal{L}^{\alpha,\beta} f &= \sum_{n=0}^{\infty} \sum_{|\kappa|=n} (-\lambda_\kappa) \hat{f}(\kappa) \vec{p}_\kappa^{\alpha,\beta} \\ T_t^{\alpha,\beta} f &= \sum_{n=0}^{\infty} \sum_{|\kappa|=n} e^{-\lambda_\kappa t} \hat{f}(\kappa) \vec{p}_\kappa^{\alpha,\beta} \\ P_t^{\alpha,\beta,\delta} f &= \sum_{n=0}^{\infty} \sum_{|\kappa|=n} e^{-\lambda_\kappa^\delta t} \hat{f}(\kappa) \vec{p}_\kappa^{\alpha,\beta}.\end{aligned}$$

As the eigenvalues λ_κ of the d -dimensional Jacobi operator do not depend linearly on $|\kappa|$, we do not have an expression of the action of $\mathcal{L}^{\alpha,\beta}$, $T_t^{\alpha,\beta}$ or $P_t^{\alpha,\beta}$ over f , in terms of the orthogonal projections over the subspaces $C_n^{\alpha,\beta}$, as in the one dimensional case (see [3]) or in the case of Hermite or Laguerre polynomial d -dimensional expansions (see [11], [9], for example).

For that reason we are going to consider, in the same spirit as the Wiener-Jacobi decomposition, an alternative decomposition of $L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$ in order to obtain expressions of $\mathcal{L}^{\alpha,\beta}$, $T_t^{\alpha,\beta}$ and $P_t^{\alpha,\beta}$ in terms of the orthogonal projections.

For fixed $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \dots, \beta_d)$, in \mathbb{R}^d such that $\alpha_i, \beta_i > -\frac{1}{2}$ let us consider the set,

$$R^{\alpha,\beta} = \left\{ r \in \mathbb{R}^+ : \text{there exists } (\kappa_1, \dots, \kappa_d) \in \mathbb{N}^d, \text{ with } r = \sum_{i=1}^d \kappa_i (\kappa_i + \alpha_i + \beta_i + 1) \right\}.$$

$R^{\alpha,\beta}$ is a numerable subset of \mathbb{R}^+ , thus it can be written as $R^{\alpha,\beta} = \{r_n\}_{n=0}^\infty$ with $r_0 < r_1 < \dots$. Let

$$A_n^{\alpha,\beta} = \left\{ \kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{N}^d : \sum_{i=1}^d \kappa_i (\kappa_i + \alpha_i + \beta_i + 1) = r_n \right\}.$$

Note that $A_0^{\alpha,\beta} = \{(0, \dots, 0)\}$ and that if $\kappa \in A_n^{\alpha,\beta}$ then $\lambda_\kappa = \sum_{i=1}^d \kappa_i (\kappa_i + \alpha_i + \beta_i + 1) = r_n$.

Let $G_n^{\alpha,\beta}$ denote the closed subspace of $L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$ generated by the linear combinations of $\{\vec{p}_\kappa^{\alpha,\beta} : \kappa \in A_n^{\alpha,\beta}\}$. By the orthogonality of the Jacobi polynomials with respect to $\mu_{\alpha,\beta}^d$ and the density of the polynomials, it is not difficult to see that $\{G_n^{\alpha,\beta}\}$ is an orthogonal decomposition of $L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$, that is

$$(9) \quad L^2([-1,1]^d, \mu_{\alpha,\beta}^d) = \bigoplus_{n=0}^{\infty} G_n^{\alpha,\beta}.$$

We call (9) a modified Wiener–Jacobi decomposition, compare with (8).

Let us denote by $J_n^{\alpha,\beta}$ the orthogonal projection of $L^2([-1, 1]^d, \mu_{\alpha,\beta}^d)$ onto $G_n^{\alpha,\beta}$. Then, for $f \in L^2([-1, 1]^d, \mu_{\alpha,\beta}^d)$ its Jacobi expansion now can be written as

$$(10) \quad f = \sum_{n=0}^{\infty} J_n^{\alpha,\beta} f$$

where

$$J_n^{\alpha,\beta} f = \sum_{\kappa \in A_n^{\alpha,\beta}} \hat{f}(\kappa) \vec{p}_\kappa^{\alpha,\beta}$$

with $\hat{f}(\kappa) = \int_{[-1,1]^d} f(x) \vec{p}_\kappa^{\alpha,\beta}(x) \mu_{\alpha,\beta}(dx)$ the Jacobi–Fourier coefficient of f for the multi-index κ .

By (3), (4), (6) we have that for $f \in L^2([-1, 1]^d, \mu_{\alpha,\beta}^d)$ with Jacobi expansion $f = \sum_{n=0}^{\infty} J_n^{\alpha,\beta} f$, the action of $\mathcal{L}^{\alpha,\beta}$, $T_t^{\alpha,\beta}$ or $P_t^{\alpha,\beta}$ over f can now be expressed as

$$(11) \quad \mathcal{L}^{\alpha,\beta} f = \sum_{n=0}^{\infty} (-r_n) J_n^{\alpha,\beta} f,$$

$$(12) \quad T_t^{\alpha,\beta} f = \sum_{n=0}^{\infty} e^{-r_n t} J_n^{\alpha,\beta} f,$$

$$(13) \quad P_t^{\alpha,\beta,\delta} f = \sum_{n=0}^{\infty} e^{-r_n^\delta t} J_n^{\alpha,\beta} f.$$

Thus using the modified Wiener–Jacobi decomposition (9) we are able to obtain expansions of $\mathcal{L}^{\alpha,\beta}$, $T_t^{\alpha,\beta}$ and $P_t^{\alpha,\beta,\delta}$ in terms of the orthogonal projections $J_n^{\alpha,\beta}$. As we have mentioned before, this can not be done with the usual Wiener–Jacobi decomposition (8).

As a consequence of the hypercontractive property of the d -dimensional Jacobi operator we have that the orthogonal projections $J_n^{\alpha,\beta}$ can be extended continuously to $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$, more formally

Proposition 2.1. *If $1 < p < \infty$ then for every $n \in \mathbb{N}$, $J_n^{\alpha,\beta}$, restricted to the polynomials \mathcal{P} , can be extended to a continuous operator to $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$, that will also be denoted as $J_n^{\alpha,\beta}$, that is, there exists $C_{n,p} \in \mathbb{R}^+$ such that*

$$\|J_n^{\alpha,\beta} f\|_p \leq C_{n,p} \|f\|_p,$$

for $f \in L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$.

Proof. First remember that the polynomials \mathcal{P} are dense in $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$, see [7]. Now let us consider $p > 2$ and for the initial condition $q(0) = 2$, let t_0 be a

positive number such that $q(t_0) = p$. Taking $f \in \mathcal{P}$, then by the hypercontractive property, Parseval's identity and Hölder's inequality we obtain,

$$\|T_{t_0}^{\alpha,\beta} J_n^{\alpha,\beta} f\|_p \leq \|J_n^{\alpha,\beta} f\|_2 \leq \|f\|_2 \leq \|f\|_p.$$

Now, as $T_{t_0}^{\alpha,\beta} J_n^{\alpha,\beta} f = e^{-t_0 r_n} J_n^{\alpha,\beta} f$ we get

$$\|J_n^{\alpha,\beta} f\|_p \leq C_{n,p} \|f\|_p,$$

with $C_{n,p} = e^{t_0 r_n}$. The general result now follows by density.

Finally, for $1 < p < 2$ the result follows by duality. \square

3. THE RESULTS

Giving a function $\Phi : \mathbb{N} \rightarrow \mathbb{R}$ the multiplier operator associated to Φ is defined as

$$(14) \quad T_\Phi f = \sum_{n=0}^{\infty} \Phi(n) J_n^{\alpha,\beta} f,$$

for $f = \sum_{n=0}^{\infty} J_n^{\alpha,\beta} f, \in \mathcal{P}$, a polynomial.

If Φ is a bounded function, then by Parseval's identity it is immediate that T_Φ is bounded on $L^2([-1, 1]^d, \mu_{\alpha,\beta}^d)$. In the case of Hermite expansions, the P.A. Meyer's Multiplier Theorem [12] gives conditions over Φ so that the multiplier T_Φ can be extended to a continuous operator on L^p for $p \neq 2$. In a previous paper [3], we have proven an analogous result for one dimensional Jacobi expansion. Now we are going to present the analogous result for d -dimensional Jacobi expansions. In order to establish this, we need some previous results.

First we note that for $n \in \mathbb{N}$, $r_n \geq n$. Then, as a consequence of the L^p continuity of the projections $J_n^{\alpha,\beta}$ and of the hypercontractivity of the d -dimensional Jacobi operator we have

Lemma 3.1. *Let $1 < p < \infty$. Then, for each $m \in \mathbb{N}$ there exists a constant C_m such that*

$$\|T_t^{\alpha,\beta} (I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta}) f\|_p \leq C_m e^{-tm} \|f\|_p.$$

Proof. Let $p > 2$ and for the initial condition $q(0) = 2$, let t_0 be a positive number such that $q(t_0) = p$.

If $t \leq t_0$, since $T_t^{\alpha,\beta}$ is a contraction, by the L^p continuity of the projections $J_n^{\alpha,\beta}$,

$$\begin{aligned} \|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f\|_p &\leq \|(I - J_0^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f\|_p \\ &\leq \|f\|_p + \sum_{n=0}^{m-1} \|J_n^{\alpha,\beta}f\|_p \\ &\leq (1 + \sum_{n=0}^{m-1} e^{t_0 r_n})\|f\|_p. \end{aligned}$$

But since $e^{t_0 r_n} \leq e^{t_0 r_m}$ for all $0 \leq n \leq m-1$ and $r_m \geq m$ for all $m \geq 1$, we get

$$\begin{aligned} \|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f\|_p &\leq (1 + m e^{t_0 r_m})\|f\|_p = C_m e^{-t_0 r_m} \|f\|_p \\ &\leq C_m e^{-tm} \|f\|_p, \end{aligned}$$

with $C_m = (1 + m e^{t_0 r_m})e^{t_0 m}$.

Now suppose $t > t_0$. For $f = \sum_{n=0}^{\infty} J_n^{\alpha,\beta} f$, by the hypercontractive property,

$$\begin{aligned} \|T_{t_0}^{\alpha,\beta} T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f\|_p^2 &\leq \|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f\|_2^2 \\ &= \|T_t^{\alpha,\beta}(\sum_{n=m}^{\infty} J_n^{\alpha,\beta} f)\|_2^2 = \|\sum_{n=m}^{\infty} e^{-tr_n} J_n^{\alpha,\beta} f\|_2^2 \\ &= \sum_{n=m}^{\infty} e^{-2tr_n} \|J_n^{\alpha,\beta} f\|_2^2 \leq \sum_{n=m}^{\infty} e^{-2tn} \|J_n^{\alpha,\beta} f\|_2^2, \end{aligned}$$

as $r_n \geq n$ for all $n \geq 1$. Then, as $m \leq n$,

$$\begin{aligned} \sum_{n=m}^{\infty} e^{-2tn} \|J_n^{\alpha,\beta} f\|_2^2 &\leq e^{-2tm} \sum_{n=0}^{\infty} \|J_{n+m}^{\alpha,\beta} f\|_2^2 \leq e^{-2tm} \sum_{n=0}^{\infty} \|J_n^{\alpha,\beta} f\|_2^2 = e^{-2tm} \|f\|_2^2 \\ &\leq e^{-2tm} \|f\|_p^2. \end{aligned}$$

Thus

$$\|T_{t_0}^{\alpha,\beta} T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f\|_p \leq e^{-tm} \|f\|_p,$$

and therefore,

$$\begin{aligned} \|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f\|_p &= \|T_{t_0}^{\alpha,\beta} T_{t-t_0}^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f\|_p \\ &\leq e^{-(t-t_0)m} \|f\|_p = C_m e^{-tm} \|f\|_p, \end{aligned}$$

with $C_m = e^{t_0 m}$. For $1 < p < 2$ the result follows by duality. \square

Using (5) and Minkowski's integral inequality, it is not difficult to see an analogous result for the generalized Poisson–Jacobi semigroup, that is, for $1 < p < \infty$ and each

$m \in \mathbb{N}$, there exists C_m such that

$$(15) \quad \|P_t^{\alpha,\beta,\gamma}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f\|_p \leq C_m e^{-tm^\gamma} \|f\|_p.$$

From the generalized Poisson–Jacobi semigroup let us define a new family of operators $\{P_{k,\gamma,m}^{\alpha,\beta}\}_{k \in \mathbb{N}}$ by the formula

$$P_{k,\gamma,m}^{\alpha,\beta} f = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} P_t^{\alpha,\beta,\gamma}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})f dt.$$

By the preceding lemma and again by Minkowski's integral inequality we have the L^p -continuity of $P_{k,\gamma,m}^{\alpha,\beta}$, for every $m \in \mathbb{N}$, that is to say, for $1 < p < \infty$ and then is a constant C_m such that

$$(16) \quad \|P_{k,\gamma,m}^{\alpha,\beta} f\|_p \leq \frac{C_m}{m^{\gamma k}} \|f\|_p.$$

In particular, if we take $n \geq m$ and $\kappa \in A_n^{\alpha,\beta}$, then

$$P_t^{\alpha,\beta,\gamma}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \dots - J_{m-1}^{\alpha,\beta})\vec{p}_\kappa^{\alpha,\beta} = e^{-r_n^\gamma t} \vec{p}_\kappa^{\alpha,\beta},$$

and thus, for all $k \in \mathbb{N}$

$$P_{k,\gamma,m}^{\alpha,\beta} \vec{p}_\kappa^{\alpha,\beta} = \frac{1}{r_n^{\gamma k}} \vec{p}_\kappa^{\alpha,\beta}.$$

Therefore, for $f \in L^2([-1, 1]^d, \mu_{\alpha,\beta}^d)$, $n \geq m$ and $k \in A_n^{\alpha,\beta}$

$$(17) \quad P_{k,\gamma,m}^{\alpha,\beta} J_n^{\alpha,\beta} f = \frac{1}{r_n^{\gamma k}} J_n^{\alpha,\beta} f$$

and if $n < m$, $k \in A_n^{\alpha,\beta}$

$$(18) \quad P_{k,\gamma,m}^{\alpha,\beta} J_n^{\alpha,\beta} f = 0.$$

We are ready to establish P.A. Meyer's Multipliers Theorem for d -dimensional Jacobi expansions.

Theorem 3.2. *If for some $n_0 \in \mathbb{N}$ and $0 < \gamma < 1$*

$$\Phi(k) = h\left(\frac{1}{r_k^\gamma}\right), \quad k \geq n_0,$$

with h an analytic function in a neighborhood of zero, then T_Φ , the multiplier operator associated to Φ , (14), admits a continuous extension to $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$.

Proof. Let

$$T_\Phi f = T_\phi^1 f + T_\Phi^2 f = \sum_{k=0}^{n_0-1} \Phi(k) J_k^{\alpha,\beta} f + \sum_{k=n_0}^{\infty} \Phi(k) J_k^{\alpha,\beta} f.$$

By Lemma 2.1 we have that

$$\|T_\Phi^1 f\|_p \leq \sum_{k=0}^{n_0-1} |\Phi(k)| \|J_k^{\alpha,\beta} f\|_p \leq \left(\sum_{k=0}^{n_0-1} |\Phi(k)| C_k \right) \|f\|_p,$$

that is, T_Φ^1 is L^p continuous. It remains to be seen that T_Φ^2 is also L^p continuous.

By hypothesis h can be written as $h(x) = \sum_{n=0}^{\infty} a_n x^n$, for x in a neighborhood of zero, then

$$T_\Phi^2 f = \sum_{k=n_0}^{\infty} \Phi(k) J_k^{\alpha,\beta} f = \sum_{k=n_0}^{\infty} h\left(\frac{1}{r_k^\gamma}\right) J_k^{\alpha,\beta} f = \sum_{k=n_0}^{\infty} \sum_{n=0}^{\infty} a_n \frac{1}{r_k^{\gamma n}} J_k^{\alpha,\beta} f,$$

but by (17) and (18), for $k \geq n_0$, $\frac{1}{r_k^{\gamma n}} J_k^{\alpha,\beta} f = P_{n,\gamma,n_0}^{\alpha,\beta} J_k^{\alpha,\beta} f$, we have

$$\begin{aligned} T_\Phi^2 f &= \sum_{k=n_0}^{\infty} \sum_{n=0}^{\infty} a_n P_{n,\gamma,n_0}^{\alpha,\beta} J_k^{\alpha,\beta} f = \sum_{n=0}^{\infty} a_n \sum_{k=n_0}^{\infty} P_{n,\gamma,n_0}^{\alpha,\beta} J_k^{\alpha,\beta} f \\ &= \sum_{n=0}^{\infty} a_n P_{n,\gamma,n_0}^{\alpha,\beta} \sum_{k=n_0}^{\infty} J_k^{\alpha,\beta} f = \sum_{n=0}^{\infty} a_n P_{n,\gamma,n_0}^{\alpha,\beta} f. \end{aligned}$$

Since $P_{n,\gamma,n_0}^{\alpha,\beta}$ is L^p continuous, (16), we obtain,

$$\begin{aligned} \|T_\Phi^2 f\|_p &\leq \sum_{n=0}^{\infty} |a_n| \|P_{n,\gamma,n_0}^{\alpha,\beta} f\|_p \\ &\leq \left(\sum_{n=0}^{\infty} |a_n| C_{n_0} \frac{1}{n_0^{\gamma n}} \right) \|f\|_p = C_{n_0} \left(\sum_{n=0}^{\infty} |a_n| \frac{1}{n_0^{\gamma n}} \right) \|f\|_p = C_{n_0} h\left(\frac{1}{n_0^\gamma}\right) \|f\|_p. \end{aligned}$$

Therefore, T_Φ is continuous in $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$. \square

As in the classical case of the Laplacian [16] and in the one dimensional Jacobi case [3], for $\gamma > 0$ we define the fractional integral of order γ , $I_\gamma^{\alpha,\beta}$, with respect to d -dimensional Jacobi operator, as

$$(19) \quad I_\gamma^{\alpha,\beta} = (-\mathcal{L}^{\alpha,\beta})^{-\gamma/2}.$$

$I_\gamma^{\alpha,\beta}$ is also called Riesz potential of order γ .

Observe that, since zero is an eigenvalue of $\mathcal{L}^{\alpha,\beta}$, then $I_\gamma^{\alpha,\beta}$ is not defined over all $L^2([-1, 1]^d, \mu_{\alpha,\beta}^d)$. Then consider $\Pi_0 = I - J_0^{\alpha,\beta}$ and denote also by $I_\gamma^{\alpha,\beta}$ the operator $(-\mathcal{L}^{\alpha,\beta})^{-\gamma/2} \Pi_0$. Then, this operator is well defined over all $L^2([-1, 1]^d, \mu_{\alpha,\beta}^d)$. In particular, for the Jacobi polynomial of order κ with $\kappa \in A_n^{\alpha,\beta}$ we have

$$(20) \quad I_\gamma^{\alpha,\beta} \vec{p}_\kappa^{\alpha,\beta} = \frac{1}{\lambda_\kappa^{\gamma/2}} \vec{p}_\kappa^{\alpha,\beta} = \frac{1}{r_n^{\gamma/2}} \vec{p}_\kappa^{\alpha,\beta}.$$

Thus, for f a polynomial in $L^2([-1, 1]^d, \mu_{\alpha, \beta}^d)$ with Jacobi expansion $\sum_{n=0}^{\infty} J_n^{\alpha, \beta} f$, we have

$$I_{\gamma}^{\alpha, \beta} f = \sum_{n=1}^{\infty} \frac{1}{r_n^{\gamma/2}} J_n^{\alpha, \beta} f.$$

Now, for $\kappa \in A_n^{\alpha, \beta}$ we have that by the change of variables $s = \lambda_{\kappa}^{1/2} t$

$$\begin{aligned} \frac{1}{\Gamma(\gamma)} \int_0^{\infty} t^{\gamma-1} P_t^{\alpha, \beta} \vec{p}_{\kappa}^{\alpha, \beta} dt &= \frac{1}{\Gamma(\gamma)} \int_0^{\infty} t^{\gamma-1} e^{-\lambda_{\kappa}^{1/2} t} dt \vec{p}_{\kappa}^{\alpha, \beta} \\ &= \frac{1}{\Gamma(\gamma)} \frac{1}{\lambda_{\kappa}^{\gamma/2}} \int_0^{\infty} s^{\gamma-1} e^{-s} ds \vec{p}_{\kappa}^{\alpha, \beta} = \frac{1}{\lambda_{\kappa}^{\gamma/2}} \vec{p}_{\kappa}^{\alpha, \beta}, \end{aligned}$$

where $P_t^{\alpha, \beta}$ is the Poisson–Jacobi semigroup.

Therefore for the fractional integral of order $\gamma > 0$ we have the integral representation,

$$(21) \quad I_{\gamma}^{\alpha, \beta} f = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} t^{\gamma-1} P_t^{\alpha, \beta} f dt,$$

for f polynomial.

As in the one dimensional case, Meyer’s multiplier theorem allows us to extend $I_{\gamma}^{\alpha, \beta}$ as a bounded operator on $L^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$.

Theorem 3.3. *The the fractional integral of order γ , $I_{\gamma}^{\alpha, \beta}$ admits a continuous extension, that it will also denoted as denote $I_{\gamma}^{\alpha, \beta}$, to $L^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$.*

Proof.

If $\gamma/2 < 1$, then $I_{\gamma}^{\alpha, \beta}$ is a multiplier with associated function

$$\Phi(k) = \frac{1}{r_k^{\gamma/2}} = h\left(\frac{1}{r_k^{\gamma/2}}\right)$$

where $h(z) = z$, which is analytic in a neighborhood of zero. Then the results follows immediately by Meyer’s theorem.

Now, if $\gamma/2 \geq 1$, let us consider $\beta \in \mathbb{R}$, $0 < \beta < 1$ and $\delta = \frac{\gamma}{2\beta}$. Then $\delta\beta = \frac{\gamma}{2}$. Let $h(z) = z^{\delta}$, which is analytic in a neighborhood of zero. Then we have

$$h\left(\frac{1}{r_k^{\beta}}\right) = \frac{1}{r_k^{\delta\beta}} = \frac{1}{r_k^{\gamma/2}} = \Phi(k).$$

Again the results follows applying Meyer’s theorem. \square

The Bessel potential of order $\gamma > 0$, $\mathcal{J}_\gamma^{\alpha,\beta}$, associated to the d -dimensional Jacobi operator is defined as

$$(22) \quad \mathcal{J}_\gamma^{\alpha,\beta} = (I - \mathcal{L}^{\alpha,\beta})^{-\gamma/2}.$$

For the Jacobi polynomial of order κ with $\kappa \in A_n^{\alpha,\beta}$ we have

$$\mathcal{J}_\gamma^{\alpha,\beta} \vec{p}_\kappa^{\alpha,\beta} = \frac{1}{(1 + \lambda_\kappa)^{\gamma/2}} \vec{p}_\kappa^{\alpha,\beta} = \frac{1}{(1 + r_n)^{\gamma/2}} \vec{p}_\kappa^{\alpha,\beta},$$

and, therefore if $f \in L^2([-1, 1]^d, \mu_{\alpha,\beta}^d)$ polynomial with expansion $\sum_{n=0}^\infty J_n^{\alpha,\beta} f$

$$(23) \quad \mathcal{J}_\gamma^{\alpha,\beta} f = \sum_{n=0}^\infty \frac{1}{(1 + r_n)^{\gamma/2}} J_n^{\alpha,\beta} f.$$

Again Meyer's theorem allows us to extend Bessel potentials to a continuous operator on $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$,

Theorem 3.4. *The operator $\mathcal{J}_\gamma^{\alpha,\beta}$ admits a continuous extension, that it will be also denoted as $\mathcal{J}_\gamma^{\alpha,\beta}$, to $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$.*

Proof. The Bessel Potential of order γ is a multiplier associated to the function $\Phi(k) = \left(\frac{1}{1+r_k}\right)^{\gamma/2}$. Let $\beta \in \mathbb{R}$, $\beta > 1$ and $h(z) = \left(\frac{z^\beta}{z^\beta+1}\right)^{\gamma/2}$. Then h is an analytic function on a neighborhood of zero and

$$h\left(\frac{1}{r_k^{1/\beta}}\right) = \left(\frac{1}{1+r_k}\right)^{\gamma/2} = \Phi(k).$$

The results follows applying Meyer's theorem. \square

Now, again by analogy to the the classical case of the Laplacian [16], we define the fractional derivative of order $\gamma > 0$, $D_\gamma^{\alpha,\beta}$, with respect to the d -dimensional Jacobi operator as

$$(24) \quad D_\gamma^{\alpha,\beta} = (-\mathcal{L}^{\alpha,\beta})^{\gamma/2}.$$

For the Jacobi polynomial of order κ with $\kappa \in A_n^{\alpha,\beta}$ we have

$$(25) \quad D_\gamma^{\alpha,\beta} \vec{p}_\kappa^{\alpha,\beta} = \lambda_\kappa^{\gamma/2} \vec{p}_\kappa^{\alpha,\beta} = r_n^{\gamma/2} \vec{p}_\kappa^{\alpha,\beta},$$

and therefore, by the density of the polynomials in $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$, $1 < p < \infty$, $D_\gamma^{\alpha,\beta}$ can be extended to $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$.

Also for the Jacobi polynomial of order κ with $\kappa \in A_n^{\alpha,\beta}$ by the change of variables $s = \lambda_\kappa^{1/2} t$,

$$\begin{aligned} \int_0^\infty t^{-\gamma-1} (P_t^{\alpha,\beta} \vec{p}_\kappa^{\alpha,\beta} - \vec{p}_\kappa^{\alpha,\beta}) dt &= \int_0^\infty t^{-\gamma-1} (e^{-\lambda_\kappa^{1/2} t} - 1) dt \vec{p}_\kappa^{\alpha,\beta} \\ &= \lambda_\kappa^{\gamma/2} \int_0^\infty s^{-\gamma-1} (e^{-s} - 1) ds \vec{p}_\kappa^{\alpha,\beta}. \end{aligned}$$

Therefore for the fractional derivative of order $0 < \gamma < 1$ we also have a integral representation,

$$(26) \quad D_\gamma^{\alpha,\beta} f = \frac{1}{c_\gamma} \int_0^\infty t^{-\gamma-1} (P_t^{\alpha,\beta} f - f) dt,$$

for f polynomial, where $c_\gamma = \int_0^\infty s^{-\gamma-1} (e^{-s} - 1) ds$.

If f is a polynomial, by (20) and (25) we have,

$$(27) \quad I_\gamma^{\alpha,\beta} (D_\gamma^{\alpha,\beta} f) = D_\gamma^{\alpha,\beta} (I_\gamma^{\alpha,\beta} f) = \Pi_0 f.$$

Let us consider now the Jacobi Sobolev spaces or potential spaces. For $1 < p < \infty$, $L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$, the Jacobi Sobolev space of order $\gamma > 0$, is defined as the completion of the set of polynomials \mathcal{P} with respect to the norm

$$\|f\|_{p,\gamma} := \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f\|_p.$$

That is to say $f \in L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ if, and only if, there is a sequence of polynomials $\{f_n\}$ such that $\lim_{n \rightarrow \infty} \|f_n - f\|_{p,\gamma} = 0$.

As in the classical case, the Jacobi Sobolev space $L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ can also be defined as the image of $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ under the Bessel Potential $\mathcal{J}_\gamma^{\alpha,\beta}$, that is,

$$L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d) = \mathcal{J}_\gamma^{\alpha,\beta} L^p([-1, 1]^d, \mu_{\alpha,\beta}^d).$$

The next proposition gives us some inclusion properties among Jacobi Sobolev spaces,

Proposition 3.5. *For the Jacobi Sobolev spaces $L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$, we have*

- i) *If $p < q$, then $L_\gamma^q([-1, 1]^d, \mu_{\alpha,\beta}^d) \subseteq L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ for each $\gamma > 0$.*
- ii) *If $0 < \gamma < \delta$, then $L_\delta^p([-1, 1]^d, \mu_{\alpha,\beta}^d) \subseteq L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ for each $0 < p < \infty$.*

Proof.

- i) For γ fixed, it follows immediately by Hölder's inequality.

ii) Let f be a polynomial and consider

$$\phi = (I - \mathcal{L}^{\alpha,\beta})^{\delta/2} f = \sum_{n=0}^{\infty} (1 + r_n)^{\delta/2} J_n^{\alpha,\beta} f,$$

which is also a polynomial. Then $\phi \in L_\delta^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$, $\|\phi\|_p = \|f\|_{p,\delta}$ and $\mathcal{J}_{(\gamma-\delta)}^{\alpha,\beta} \phi = (I - \mathcal{L}^{\alpha,\beta})^{(\gamma-\delta)/2} \phi = (I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f$, by the L^p -continuity of Bessel Potentials,

$$\|f\|_{p,\gamma} = \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f\|_p = \|\mathcal{J}_{(\gamma-\delta)}^{\alpha,\beta} \phi\|_p \leq C_p \|f\|_{p,\delta}.$$

Now let $f \in L_\delta^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$. Then there exists $g \in L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ such that $f = \mathcal{J}_\delta^{\alpha,\beta} g$ and a sequence of polynomials $\{g_n\}$ in $L^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ such that $\lim_{n \rightarrow \infty} \|g_n - g\|_p = 0$. Set $f_n = \mathcal{J}_\delta^{\alpha,\beta} g_n$. Then $\lim_{n \rightarrow \infty} \|f_n - f\|_{p,\delta} = 0$, and

$$\begin{aligned} \|f_n - f\|_{p,\gamma} &= \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} (f_n - f)\|_p = \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} (I - \mathcal{L}^{\alpha,\beta})^{-\delta/2} (g_n - g)\|_p \\ &= \|(I - \mathcal{L}^{\alpha,\beta})^{(\gamma-\delta)/2} (g_n - g)\|_p = \|\mathcal{J}_{(\gamma-\delta)}^{\alpha,\beta} (g_n - g)\|_p, \end{aligned}$$

by the L^p continuity of Bessel Potentials $\lim_{n \rightarrow \infty} \|f_n - f\|_{p,\gamma} = 0$.

Therefore,

$$\begin{aligned} \|f\|_{p,\gamma} &\leq \|f_n - f\|_{p,\gamma} + \|f_n\|_{p,\gamma} \\ &\leq \|f_n - f\|_{p,\gamma} + \|f_n\|_{p,\delta}, \end{aligned}$$

taking limit as n goes to infinity, we obtain the result. \square

Let us consider the space

$$L_\gamma([-1, 1]^d, \mu_{\alpha,\beta}^d) = \bigcup_{p>1} L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d).$$

$L_\gamma([-1, 1]^d, \mu_{\alpha,\beta}^d)$ is the natural domain of $D_\gamma^{\alpha,\beta}$. We define it in this space as follows.

Let $f \in L_\gamma([-1, 1]^d, \mu_{\alpha,\beta}^d)$, then there is $p > 1$ such that $f \in L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$ and a sequence $\{f_n\}$ polynomials such that $\lim_{n \rightarrow \infty} f_n = f$ in $L_\gamma^p([-1, 1]^d, \mu_{\alpha,\beta}^d)$. We define for $f \in L_\gamma([-1, 1]^d, \mu_{\alpha,\beta}^d)$

$$D_\gamma^{\alpha,\beta} f = \lim_{n \rightarrow \infty} D_\gamma^{\alpha,\beta} f_n.$$

The next theorem shows that $D_\gamma^{\alpha,\beta}$ is well defined and also inequality (29) gives us a characterization of the Sobolev spaces,

Theorem 3.6. *Let $\gamma > 0$ and $1 < p, q < \infty$.*

i) *If $\{f_n\}$ is a sequence of polynomials such that*

$$\lim_{n \rightarrow \infty} f_n = f$$

in $L_\gamma^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$, then

$$(28) \quad \lim_{n \rightarrow \infty} D_\gamma^{\alpha, \beta} f_n \in L^p([-1, 1]^d, \mu_{\alpha, \beta}^d),$$

and the limit does not depend on the choice of the sequence $\{f_n\}$.

If $f \in L_\gamma^p([-1, 1]^d, \mu_{\alpha, \beta}^d) \cap L_\gamma^q([-1, 1]^d, \mu_{\alpha, \beta}^d)$, then the limit does not depend on the choice of p or q . Thus $D_\gamma^{\alpha, \beta}$ is well defined on $L_\gamma([-1, 1]^d, \mu_{\alpha, \beta}^d)$.

- ii) $f \in L_\gamma^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$ if, and only if, $D_\gamma^{\alpha, \beta} f \in L^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$. Moreover, there exists positive constants $A_{p, \gamma}$ and $B_{p, \gamma}$ such that

$$(29) \quad B_{p, \gamma} \|f\|_{p, \gamma} \leq \|D_\gamma^{\alpha, \beta} f\|_p \leq A_{p, \gamma} \|f\|_{p, \gamma}.$$

Proof.

ii) First, let us note that for $f = \sum_{n=0}^{\infty} J_n^{\alpha, \beta} f$ polynomial,

$$D_\gamma^{\alpha, \beta} \mathcal{J}_\gamma^{\alpha, \beta} f = \sum_{n=0}^{\infty} \left(\frac{r_n}{1+r_n} \right)^{\gamma/2} J_n^{\alpha, \beta} f,$$

that is, the operator $D_\gamma^{\alpha, \beta} \mathcal{J}_\gamma^{\alpha, \beta}$ is a multiplier with associated function $\Phi(k) = \left(\frac{r_k}{1+r_k} \right)^{\gamma/2} = h(\frac{1}{r_k})$ where $h(z) = \left(\frac{1}{z+1} \right)^{\gamma/2}$, and therefore by Meyer's theorem it is L^p -continuous.

Let f be a polynomial and let ϕ be a polynomial such that $f = \mathcal{J}_\gamma^{\alpha, \beta} \phi$. We have that $\|f\|_{p, \gamma} = \|\phi\|_p$ and by the continuity of the operator $D_\gamma^{\alpha, \beta} \mathcal{J}_\gamma^{\alpha, \beta}$

$$\|D_\gamma^{\alpha, \beta} f\|_p = \|D_\gamma^{\alpha, \beta} \mathcal{J}_\gamma^{\alpha, \beta} \phi\|_p \leq A_{p, \gamma} \|\phi\|_p = A_{p, \gamma} \|f\|_{p, \gamma}.$$

To prove the converse, let us suppose that f polynomial, then $D_\gamma^{\alpha, \beta} f$ is also a polynomial, and therefore $D_\gamma^{\alpha, \beta} f \in L^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$. Consider

$$\phi = (I - \mathcal{L}^{\alpha, \beta})^{\gamma/2} f = \sum_{k=0}^{\infty} (1+r_k)^{\gamma/2} J_k^{\alpha, \beta} f = \sum_{k=0}^{\infty} \left(\frac{1+r_k}{r_k} \right)^{\gamma/2} J_k^{\alpha, \beta} (D_\gamma^{\alpha, \beta} f).$$

The mapping

$$g = \sum_{k=0}^{\infty} J_k^{\alpha, \beta} g \mapsto \sum_{k=0}^{\infty} \left(\frac{1+r_k}{r_k} \right)^{\gamma/2} J_k^{\alpha, \beta} g$$

is a multiplier with associated function $\Phi(k) = \left(\frac{1+r_k}{r_k} \right)^{\gamma/2} = h(\frac{1}{r_k})$ where $h(z) = (z+1)^{\gamma/2}$, so by Meyer's theorem, taking $g = D_\gamma^{\alpha, \beta} f$ we have

$$\|f\|_{p, \gamma} = \|\phi\|_p \leq B_{p, \gamma} \|D_\gamma^{\alpha, \beta} f\|_p.$$

Thus we get (29) for polynomials.

For the general case, $f \in L_\gamma^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$, then there exists $g \in L^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$ such that $f = \mathcal{J}_\gamma^{\alpha, \beta} g$ and a sequence $\{g_n\}$ of polynomials such that $\lim_{n \rightarrow \infty} \|g_n - g\|_p = 0$. Let $f_n = \mathcal{J}_\gamma^{\alpha, \beta} g_n$, then $\lim_{n \rightarrow \infty} \|f_n - f\|_{p, \gamma} = 0$. Then, by the continuity of the operator $D_\gamma^{\alpha, \beta} \mathcal{J}_\gamma^{\alpha, \beta}$ and as $\lim_{n \rightarrow \infty} \|g_n - g\|_p = 0$,

$$\lim_{n \rightarrow \infty} \|D_\gamma^{\alpha, \beta}(f_n - f)\|_p = \lim_{n \rightarrow \infty} \|D_\gamma^{\alpha, \beta} \mathcal{J}_\gamma^{\alpha, \beta}(g_n - g)\|_p = 0.$$

Then, as

$$B_{p, \gamma} \|f_n\|_{p, \gamma} \leq \|D_\gamma^{\alpha, \beta} f_n\|_p \leq A_{p, \gamma} \|f_n\|_{p, \gamma},$$

the results follows by taking the limit as $n \rightarrow \infty$ in this inequality.

i) Let $\{f_n\}$ be a sequence of polynomials such that

$$\lim_{n \rightarrow \infty} f_n = f,$$

in $L_\gamma^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$. Then, by (29)

$$\lim_{n \rightarrow \infty} \|D_\gamma^{\alpha, \beta} f_n\|_p \leq B_{p, \gamma} \lim_{n \rightarrow \infty} \|f_n\|_{p, \gamma} = B_{p, \gamma} \|f\|_{p, \gamma},$$

hence, $\lim_{n \rightarrow \infty} D_\gamma^{\alpha, \beta} f_n \in L^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$.

Now suppose that $\{q_n\}$ is another sequence of polynomials such that $\lim_{n \rightarrow \infty} q_n = f$ in $L_\gamma^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$. Then $\lim_{n \rightarrow \infty} f_n - q_n = 0$. By (29)

$$B_{p, \gamma} \|f_n - q_n\|_{p, \gamma} \leq \|D_\gamma^{\alpha, \beta} f_n - D_\gamma^{\alpha, \beta} q_n\|_p \leq A_{p, \gamma} \|f_n - q_n\|_{p, \gamma},$$

and now, taking the limit as $n \rightarrow \infty$ we get that $\lim_{n \rightarrow \infty} D_\gamma^{\alpha, \beta} f_n = \lim_{n \rightarrow \infty} D_\gamma^{\alpha, \beta} q_n$ in $L^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$ and therefore the limit does not depends on the choice of the approximating sequence.

Finally, let us suppose that $f \in L_\gamma^p([-1, 1]^d, \mu_{\alpha, \beta}^d) \cap L_\gamma^q([-1, 1]^d, \mu_{\alpha, \beta}^d)$ and, without loss of generality, let us assume that $p \leq q$, then by Proposition 3.5, i), $L_\gamma^q([-1, 1]^d, \mu_{\alpha, \beta}^d) \subseteq L_\gamma^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$ and therefore $f \in L_\gamma^q([-1, 1]^d, \mu_{\alpha, \beta}^d)$. Now, if $\{f_n\}$ is a sequence of polynomials such that $\lim_{n \rightarrow \infty} f_n = f$ in $L_\gamma^q([-1, 1]^d, \mu_{\alpha, \beta}^d)$ (hence in $L_\gamma^p([-1, 1]^d, \mu_{\alpha, \beta}^d)$), we have

$$\lim_{n \rightarrow \infty} D_\gamma^{\alpha, \beta} f_n \in L^q([-1, 1]^d, \mu_{\alpha, \beta}^d) = L^p([-1, 1]^d, \mu_{\alpha, \beta}^d) \cap L^q([-1, 1]^d, \mu_{\alpha, \beta}^d).$$

Therefore the limit does not depends on the choice of p or q . \square

In what follows we will give an alternative representation of $D_\gamma^{\alpha, \beta}$ and $I_\gamma^{\alpha, \beta}$, but first we present a technical Lemma, were we study the asymptotic behavior of the d -dimensional Poisson-Jacobi semigroup $\{P_t^{\alpha, \beta}\}$.

Lemma 3.7. *If $f \in C^2([-1, 1]^d)$ such that $\int_{[-1, 1]^d} f(y) \mu_{\alpha, \beta}^d(dy) = 0$ then*

$$(30) \quad \left| \frac{\partial}{\partial t} P_t^{\alpha, \beta} f(x) \right| \leq C_{f, \alpha, \beta, d} (1 + |x|) e^{-d_{\alpha, \beta}^{1/2} t}$$

where $d_{\alpha, \beta} = \max\{\alpha_j + \beta_j + 2 : j = 1, \dots, d\}$ and $|x|$ denotes the usual euclidian norm for $x \in \mathbb{R}^d$.

As a consequence the Poisson-Jacobi semigroup $\{P_t^{\alpha, \beta}\}_{t \geq 0}$, has exponential decay on $(C_0^{\alpha, \beta})^\perp = \bigoplus_{n=1}^\infty C_n^{\alpha, \beta}$. More explicitly, if $f \in C^2([-1, 1]^d)$, such that $\int_{[-1, 1]^d} f(y) \mu_{\alpha, \beta}^d(dy) = 0$ then

$$(31) \quad |P_t^{\alpha, \beta} f(x)| \leq C_{f, \alpha, \beta, d} (1 + |x|) e^{-d_{\alpha, \beta}^{1/2} t}.$$

Proof. First, let us see that $\left| \frac{\partial}{\partial t} T_t^{\alpha, \beta} f(x) \right| \leq C_{f, \alpha, \beta, d} (1 + |x|) e^{-d_{\alpha, \beta} t}$. Since

$$\begin{aligned} \frac{\partial}{\partial t} T_t^{\alpha, \beta} f &= \mathcal{L}^{\alpha, \beta} T_t^{\alpha, \beta} f = \sum_{i=1}^d \left[(1 - x_i^2) \frac{\partial^2}{\partial x_i^2} T_t^{\alpha, \beta} f \right. \\ &\quad \left. + (\beta_i - \alpha_i + 1 - (\alpha_i + \beta_i + 2) x_i) \frac{\partial}{\partial x_i} T_t^{\alpha, \beta} f \right] \end{aligned}$$

it is sufficient to study $\frac{\partial}{\partial x_i} T_t^{\alpha, \beta} f$ and $\frac{\partial^2}{\partial x_i^2} T_t^{\alpha, \beta} f$.

First note that for the one dimensional Jacobi polinomial $p_n^{\alpha, \beta}$, $n \in \mathbb{N}$ and the one dimensional Jacobi semigroup $T_t^{\alpha, \beta}$, $\alpha, \beta > -\frac{1}{2}$, we have

$$\begin{aligned} \frac{\partial}{\partial x} T_t^{\alpha, \beta} f &= e^{-(\alpha + \beta + 2)t} T_t^{\alpha + 1, \beta + 1} \left(\frac{\partial f}{\partial x} \right) \\ \frac{\partial^2}{\partial x^2} T_t^{\alpha, \beta} f &= e^{-2(\alpha + \beta + 3)t} T_t^{\alpha + 2, \beta + 2} \left(\frac{\partial^2 f}{\partial x^2} \right) \end{aligned}$$

therefore, as for $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^d$, $\alpha, \beta \in [-\frac{1}{2}, \infty)^d$

$$T_t^{\alpha, \beta} \vec{p}_\kappa^{\alpha, \beta} = \prod_{i=1}^d T_t^{\alpha_i, \beta_i} p_{\kappa_i}^{\alpha_i, \beta_i},$$

$$\frac{\partial}{\partial x_j} T_t^{\alpha, \beta} \vec{p}_\kappa^{\alpha, \beta}(x) = e^{-(\alpha_j + \beta_j + 2)t} T_t^{\alpha + e_j, \beta + e_j} \left(\frac{\partial}{\partial x_j} \vec{p}_\kappa^{\alpha, \beta}(x) \right)$$

and

$$\frac{\partial^2}{\partial x_j^2} T_t^{\alpha, \beta} \vec{p}_\kappa^{\alpha, \beta}(x) = e^{-2(\alpha_j + \beta_j + 3)t} T_t^{\alpha + 2e_j, \beta + 2e_j} \left(\frac{\partial^2}{\partial x_j^2} \vec{p}_\kappa^{\alpha, \beta}(x) \right)$$

with $e_j \in \mathbb{R}^d$ has one in the j -th coordinate and zero elsewhere. Then for $f \in L^2([-1, 1]^d, \mu_{\alpha, \beta}^d)$

$$\frac{\partial}{\partial x_j} T_t^{\alpha, \beta} f(x) = e^{-(\alpha_j + \beta_j + 2)t} T_t^{\alpha + e_j, \beta + e_j} \left(\frac{\partial}{\partial x_j} f(x) \right)$$

and

$$\frac{\partial^2}{\partial x_j^2} T_t^{\alpha, \beta} f(x) = e^{-2(\alpha_j + \beta_j + 3)t} T_t^{\alpha + 2e_j, \beta + 2e_j} \left(\frac{\partial^2}{\partial x_j^2} f(x) \right).$$

Hence

$$\begin{aligned} \left| \frac{\partial}{\partial t} T_t^{\alpha, \beta} f(x) \right| &= \left| \mathcal{L}^{\alpha, \beta} T_t^{\alpha, \beta} f(x) \right| \\ &\leq \sum_{j=1}^d \left[|1 - x_j^2| e^{-2(\alpha_j + \beta_j + 3)t} T_t^{\alpha + 2e_j, \beta + 2e_j} \left(\left| \frac{\partial^2}{\partial x_j^2} f(x) \right| \right) \right. \\ &\quad \left. + (|\beta_j - \alpha_j + 1| + (\alpha_j + \beta_j + 2)|x_j|) e^{-(\alpha_j + \beta_j + 2)t} T_t^{\alpha + e_j, \beta + e_j} \left(\left| \frac{\partial}{\partial x_j} f(x) \right| \right) \right]. \end{aligned}$$

As $f \in C^2([-1, 1]^d)$, there exists C_f such that $\left| \frac{\partial}{\partial x_j} f(x) \right| \leq C_f$ and $\left| \frac{\partial^2}{\partial x_j^2} f(x) \right| \leq C_f$.

Also, for each $j = 1, \dots, d$

$$\begin{aligned} |1 - x_j^2| &\leq |1 - x_j||1 + x_j| \leq 1 + |x_j| \leq 1 + |x|, \\ e^{-2(\alpha_j + \beta_j + 3)t} &\leq e^{-(\alpha_j + \beta_j + 2)t} \end{aligned}$$

$$|\beta_j - \alpha_j + 1| + (\alpha_j + \beta_j + 2)|x_j| \leq C_{\alpha, \beta}(1 + |x_j|) \leq C_{\alpha, \beta}(1 + |x|),$$

and then

$$\begin{aligned} \left| \frac{\partial}{\partial t} T_t^{\alpha, \beta} f(x) \right| &= \left| \mathcal{L}^{\alpha, \beta} T_t^{\alpha, \beta} f(x) \right| \leq C_{f, \alpha, \beta}(1 + |x|) \sum_{j=1}^d e^{-(\alpha_j + \beta_j + 2)t} \\ &\leq C_{f, \alpha, \beta, d}(1 + |x|) e^{-d_{\alpha, \beta} t}. \end{aligned}$$

Now

$$\frac{\partial}{\partial t} P_t^{\alpha, \beta} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{2u} \mathcal{L}^{\alpha, \beta} T_{t^2/4u}^{\alpha, \beta} f du,$$

then, by the change of variable $u = d_{\alpha, \beta} s$

$$\begin{aligned} \left| \frac{\partial}{\partial t} P_t^{\alpha, \beta} f(x) \right| &\leq C_{f, \alpha, \beta, d}(1 + |x|) \int_0^\infty e^{-u} \frac{t}{2\sqrt{\pi}} u^{-3/2} e^{-d_{\alpha, \beta} t^2/4u} du \\ &= C_{f, \alpha, \beta, d}(1 + |x|) \int_0^\infty e^{-d_{\alpha, \beta} s} \frac{t}{2\sqrt{\pi}} s^{-3/2} e^{-t^2/4s} ds \\ &= C_{f, \alpha, \beta, d}(1 + |x|) \int_0^\infty e^{-d_{\alpha, \beta} s} \mu_t^{1/2}(ds) = C_{f, \alpha, \beta, d}(1 + |x|) e^{-d_{\alpha, \beta}^{1/2} t}. \end{aligned}$$

Since we are assuming that $\int_{[-1,1]^d} f(y) \mu_{\alpha,\beta}^d(dy) = 0$,

$$(32) \quad \lim_{t \rightarrow \infty} P_t^{\alpha,\beta} f(x) = 0,$$

we get

$$\begin{aligned} \left| P_t^{\alpha,\beta} f(x) \right| &\leq \int_t^\infty \left| \frac{\partial}{\partial s} P_s^{\alpha,\beta} f(x) \right| ds \leq C_{f,\alpha,\beta,d} \int_t^\infty (1 + |x|) e^{-d_{\alpha,\beta}^{1/2} s} ds \\ &= C_{f,\alpha,\beta,d} (1 + |x|) e^{-d_{\alpha,\beta}^{1/2} t}. \end{aligned}$$

□

Now, since $\{P_t^{\alpha,\beta}\}_{t \geq 0}$ is an strongly continuos semigroup, we have

$$(33) \quad \lim_{t \rightarrow 0^+} P_t^{\alpha,\beta} f(x) = f(x)$$

Let us write

$$\begin{aligned} P_t^{\alpha,\beta} f(x) &= \int_0^\infty T_s^{\alpha,\beta} f(x) \mu_t^{1/2}(ds) = \int_{[-1,1]^d} \left[\int_0^\infty p_d^{\alpha,\beta}(s, x, y) \mu_t^{1/2}(ds) \right] f(y) \mu_{\alpha,\beta}^d(dy) \\ &= \int_{[-1,1]^d} k_d^{\alpha,\beta}(t, x, y) f(y) \mu_{\alpha,\beta}^d(dy), \end{aligned}$$

where

$$(34) \quad k_d^{\alpha,\beta}(t, x, y) = \int_0^\infty p_d^{\alpha,\beta}(s, x, y) \mu_t^{1/2}(ds).$$

and define the operator $Q_t^{\alpha,\beta}$ as

$$(35) \quad Q_t^{\alpha,\beta} f(x) = -t \frac{\partial}{\partial t} P_t f(x) = \int_{[-1,1]^d} q_d^{\alpha,\beta}(t, x, y) f(y) \mu_{\alpha,\beta}^d(dy),$$

with $q_d^{\alpha,\beta}(t, x, y) = -t \frac{\partial}{\partial t} k_d^{\alpha,\beta}(t, x, y)$.

Now we give the alternative representations for $D_\gamma^{\alpha,\beta}$ and $I_\gamma^{\alpha,\beta}$.

Proposition 3.8. *Suppose f differentiable with continuos derivatives up to the second order such that $\int_{[-1,1]^d} f(y) \mu_{\alpha,\beta}^d(dy) = 0$, then we have*

$$(36) \quad -\gamma D_\gamma^{\alpha,\beta} f = \frac{1}{c_\gamma} \int_0^\infty t^{-\gamma-1} Q_t^{\alpha,\beta} f dt, \quad 0 < \gamma < 1,$$

$$(37) \quad \gamma I_\gamma^{\alpha,\beta} f = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} Q_t^{\alpha,\beta} f dt, \quad \gamma > 0.$$

Proof. Let us start by proving (36). Integrating by parts in (26) we have

$$\begin{aligned}
D_\gamma^{\alpha,\beta} f &= \frac{1}{c_\gamma} \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} \int_a^b t^{\gamma-1} (P_t^{\alpha,\beta} f - f) dt \\
&= \frac{1}{c_\gamma} \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} \left[-\frac{P_t^{\alpha,\beta} f - f}{\gamma t^\gamma} \Big|_a^b + \frac{1}{\gamma} \int_a^b t^{-\gamma} \frac{\partial}{\partial t} P_t^{\alpha,\beta} f dt \right] \\
&= -\frac{1}{\gamma c_\gamma} \int_0^\infty t^{-\gamma-1} Q_t^{\alpha,\beta} f dt,
\end{aligned}$$

since, by (30),

$$\begin{aligned}
\lim_{a \rightarrow 0^+} \left| \frac{P_a^{\alpha,\beta} f(x) - f(x)}{a^\gamma} \right| &\leq \lim_{a \rightarrow 0^+} \frac{1}{a^\gamma} \int_0^a \left| \frac{\partial}{\partial s} P_s^{\alpha,\beta} f(x) \right| ds \\
&\leq C_{f,\alpha,\beta,d} (1 + |x|) \lim_{a \rightarrow 0^+} \frac{1 - e^{-d_{\alpha,\beta}^{1/2} a}}{a^\gamma} = 0
\end{aligned}$$

and by (33)

$$\lim_{b \rightarrow \infty} \frac{P_b^{\alpha,\beta} f - f}{b^\gamma} = 0.$$

Let us prove (37). Integrating by parts in (21)

$$\begin{aligned}
I_\gamma^{\alpha,\beta} f &= \frac{1}{\Gamma(\gamma)} \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} \int_a^b t^{\gamma-1} P_t^{\alpha,\beta} f dt \\
&= \frac{1}{\Gamma(\gamma)} \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} \left[\frac{t^\gamma}{\gamma} P_t^{\alpha,\beta} f \Big|_a^b - \frac{1}{\gamma} \int_a^b t^\gamma \frac{\partial}{\partial t} P_t^{\alpha,\beta} f dt \right] \\
&= \frac{1}{\gamma \Gamma(\gamma)} \int_0^\infty t^{\gamma-1} Q_t^{\alpha,\beta} f dt,
\end{aligned}$$

since, by (30)

$$\lim_{b \rightarrow \infty} \left| b^\gamma P_b^{\alpha,\beta} f \right| \leq C_{f,\alpha,\beta,d} (1 + |x|) \lim_{b \rightarrow \infty} b^\gamma e^{-d_{\alpha,\beta}^{1/2} b} = 0$$

and

$$\lim_{a \rightarrow 0^+} a^\gamma P_a^{\alpha,\beta} f = 0.$$

□

This representations for $I_\gamma^{\alpha,\beta}$ and $D_\gamma^{\alpha,\beta}$ allows us to obtain a version of Calderón's reproduction formula for the d -dimensional Jacobi measure,

Theorem 3.9. *i) Suppose $f \in L^1([-1, 1]^d, \mu_{\alpha, \beta}^d)$ such that $\int_{[-1, 1]^d} f(y) \mu_{\alpha, \beta}^d(dy) = 0$, then we have*

$$(38) \quad f = \int_0^\infty Q_t^{\alpha, \beta} f \frac{dt}{t}.$$

ii) Suppose f a polynomial such that $\int_{[-1, 1]^d} f(y) \mu_{\alpha, \beta}^d(dy) = 0$, then we have

$$(39) \quad f = C_\gamma \int_0^\infty \int_0^\infty t^{-\gamma} s^\gamma Q_t^{\alpha, \beta} (Q_s^{\alpha, \beta} f) \frac{ds}{s} \frac{dt}{t}, \quad 0 < \gamma < 1.$$

Also,

$$(40) \quad \int_0^\infty \int_0^\infty t^{-\gamma} s^\gamma Q_t^{\alpha, \beta} (Q_s^{\alpha, \beta} f) \frac{ds}{s} \frac{dt}{t} = \int_0^\infty u \frac{\partial^2}{\partial u^2} P_u^{\alpha, \beta} f du.$$

Formula (39) is the version of Calderón's reproduction formula for the d -dimensional Jacobi measure.

Proof. To prove (38) note that by (32) and (33) we have,

$$\int_0^\infty Q_t^{\alpha, \beta} f \frac{dt}{t} = \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} (- \int_a^b \frac{\partial}{\partial t} P_t^{\alpha, \beta} f dt) = \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} (-P_t^{\alpha, \beta} f)|_a^b = f.$$

Let us prove (39). Given f a polynomial such that $\int_{[-1, 1]^d} f(y) \mu_{\alpha, \beta}^d(dy) = 0$, by Proposition 3.8, we have

$$(41) \quad D_\gamma^{\alpha, \beta} (I_\gamma^{\alpha, \beta} f) = -\frac{1}{\gamma c_\gamma} \int_0^\infty t^{-\gamma-1} Q_t^{\alpha, \beta} (I_\gamma^{\alpha, \beta} f) dt.$$

Now, by (37) and the linearity of $Q_t^{\alpha, \beta}$, we have

$$Q_t^{\alpha, \beta} (I_\gamma^{\alpha, \beta} f) = \frac{1}{\gamma \Gamma(\gamma)} \int_0^\infty s^{\gamma-1} Q_t^{\alpha, \beta} (Q_s^{\alpha, \beta} f)(y) ds.$$

Substituting in (41)

$$f = D_\gamma^{\alpha, \beta} (I_\gamma^{\alpha, \beta} f) = C_\gamma \int_0^\infty \int_0^\infty t^{-\gamma-1} s^{\gamma-1} Q_t^{\alpha, \beta} (Q_s^{\alpha, \beta} f) ds dt,$$

with $C_\gamma = -\frac{1}{\gamma^2 c_\gamma \Gamma(\gamma)}$.

In order to prove (40), integrating by parts, and by Proposition 3.8 we have

$$\begin{aligned} \int_0^\infty u \frac{\partial^2}{\partial u^2} P_u^{\alpha, \beta} f(x) du &= u \frac{\partial}{\partial u} P_u^{\alpha, \beta} f(x) \Big|_0^\infty - \int_0^\infty \frac{\partial}{\partial u} P_u^{\alpha, \beta} f(x) du \\ &= - \int_0^\infty \frac{\partial}{\partial u} P_u^{\alpha, \beta} f(x) du = -P_u^{\alpha, \beta} f(x) \Big|_0^\infty \\ &= P_0^{\alpha, \beta} f(x) = f(x). \end{aligned}$$

□

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